SEARCH PRIORITIES FOR A TARGET PROBABILITY AREA

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NAVAL POSTGRADUATE SCHOOL
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THESIS

SEARCH PRIORITIES FOR A
TARGET PROBABILITY AREA

by

Patricia Ann Tracey

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#20 - ABSTRACT - (CONTINUED)

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Search Priorities for a Target Probability Area

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ABSTRACT

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I. INTRODUCTION

As long range surface launched weapons systems continue to be introduced into the fleet, the operational commander is increasingly faced with the problem of being able to launch weapons at targets located beyond the horizon. Successful employment of such weapons is dependent not only on the ability to detect, classify and localize targets at considerable distances, but also on the ability to distinguish the true target from a potentially larger field of false targets. While long range ocean surveillance sensors may be of assistance in the identification and localization of targets, the information provided may not be refined sufficiently to permit effective targeting of long range weapons on that basis alone. The on-scene commander must in general rely on additional data on target location gathered locally and close to the time of weapons launch for accurate targeting. Thus, he must still be able to detect and track the desired target and be able to distinguish it from other targets within range of his sensors.

The procedures developed in this paper are designed to be of assistance in addressing the last of these problems. They are applicable when the information available is an error ellipse around a threat location estimated by an ocean surveillance sensor and bearings only data generated by a local surface-based sensor. The question of whether a target

detected by the local sensor is the same as that whose estimated position was provided by an external sensor can only be addressed if information is available on the locations and tracks of all possible targets within range of the local sensor. Since such data is generally not available, this paper does not attempt to answer that question, but rather develops a method by which bearing information from different sensors can be compared as to the likelihood of each bearing being associated with the threat identified previously. It is envisioned that these likelihoods can be then used to induce an ordering among bearing data gathered by different sensors or, conceivably, conflicting data gathered by one sensor. The ordering would be based on the likelihood that each bearing will contribute to refining the original estimate of the location of the target of interest. This information could be applied in a number of ways: as a guide to allocation of more capable sensor resources for purposes of obtaining targeting information; as a guide for allocation of weapons against more than one threat; as a means of pre-processing data before entering it into a target motion model, thereby reducing the chance of introducing unrelated data.

To determine the likelihood that a given line of bearing and the threat coincide, consideration was given to the uncertainties inherent in estimation of target position, in the measurement of bearings by a particular sensor and in estimation of sensor location. It is assumed that at some

time t_o, an ocean surveillance sensor detects a threat whose position is estimated to be within an elliptical region with $p_1 \times 100\%$ certainty. The estimated position data are received and converted by the on-scene commander into a probability distribution described by a truncated bivariate normal density function.

It is further assumed that the standard error σ_β characteristic of the local sensor is known. The sensor bearing β with bearing error σ_β is then projected from the sensor position through the threat density function.

Since sensor position relative to the target may itself be subject to navigation error, the uncertainty is introduced as a truncated circular bivariate normal distribution centered at location (u_o, v_o) with standard deviation σ.

A TI-59 calculator program is developed which estimates the likelihood that the threat identified by an external sensor lies along bearing β, given the threat distribution, the bearing error, and sensor position distribution relative to the threat.

The theoretical basis for this calculation is presented in Chapter II. The algorithms used in designing the calculator program are described in Chapter III. A program listing and verbal flow are provided in Appendix A along with instructions for the user. Appendix B contains a verbal flow of a program designed for use when considerable time has elapsed since the initial estimate of the location of a moving target.

II. THEORETICAL BASES

The general approach to determining the likelihood that a measured line of bearing β is the true bearing from the sensor to the threat identified and localized by an external sensor is discussed in this chapter. Calculations required when using threat position information both as initially generated by the external sensor at time zero, t_o, and as distorted to account for an intervening time late, t_L, are discussed. Uncertainties in bearing measurement and sensor position are included.

Initially, assume that sensor position is known with certainty. Let β_T be the true bearing of the threat from the sensor. Since threat location is uncertain, β_T is a random quantity with probability density function $f_{\beta_{TRUE}}(\beta_T)$. Let $f_\beta(\beta;\beta_T)d\beta$ be the probability that the errors in bearing measurement are such as to give rise to a bearing on the threat in the interval $d\beta$ about the observed value β when the true bearing is β_T. Then the likelihood of observing a bearing β is:

$$f'(\beta) = \int_{\text{all } \beta_T} f_\beta(\beta;\beta_T)\, f_{\beta_{TRUE}}(\beta_T)\, d\beta_T \quad . \qquad (1)$$

The density $f_{\beta_{TRUE}}(\beta_T)$ is determined by the probability density function of threat location at time t, $f_{X,Y}(x,y;t)$.

Given the probability density function of threat location at time t and assuming that bearing errors are normally distributed with mean zero, consideration is limited to the probability that the observed bearing is the true bearing of the threat, given the threat is contained in a planar region E and the true bearing β_T is in an interval about β with upper and lower bounds β_U and β_L. The region E is selected to be the minimum area planar region which contains the threat with a specified high probability p_1. The interval (β_L, β_U) is selected so that, for all the possible values of β_T contained in the interval, β is within an interval of specified high probability p_2 around β_T. The fan (β_L, β_U) is symmetric about β so that it represents the minimum area region which meets the above criterion. The likelihood of observing β when the threat is in E and β_T is in (β_L, β_U) is determined by integrating expression (1) above, over a region defined in the manner of the shaded area of Figure 1(a).

If sensor position is not certain, assume that it is distributed in accordance with a circular bivariate normal distribution $f_{U,V}(u,v)$. Again, consideration is limited to determining the likelihood of observing a bearing β given that the threat is contained in a planar region, E, β_T is in (β_L, β_U), and the sensor is contained in a planar region C. The region C is selected as the minimum area planar region which contains the sensor with a high probability p_3. The region of E over which $f'(\beta)$ is evaluated expands as the shaded region of Figure 1(b).

Given the probability density function of threat location

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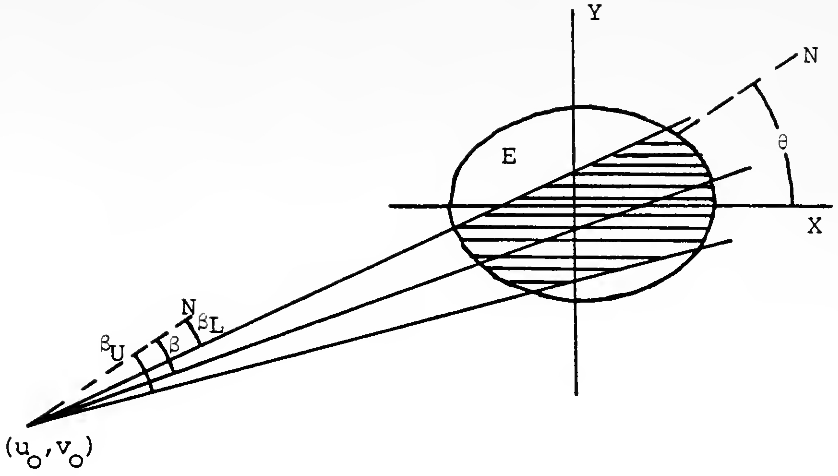
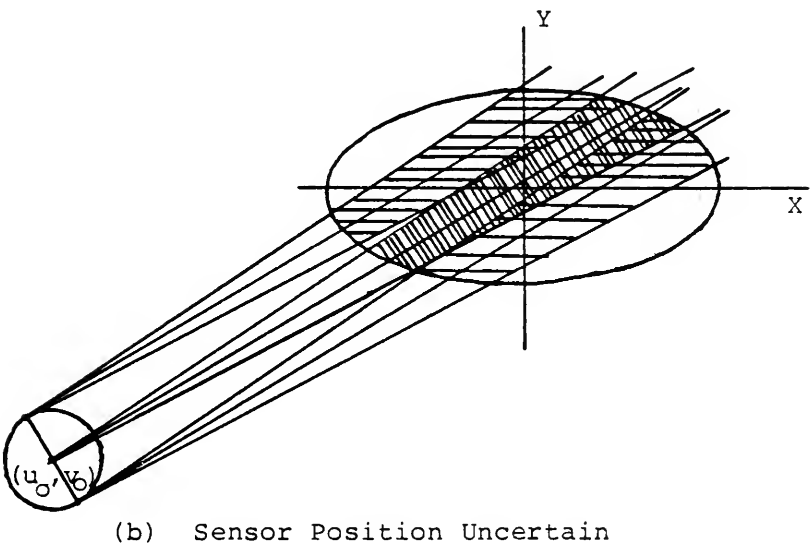


(a) Sensor Position Certain



(b) Sensor Position Uncertain

FIGURE 1. REGION OF INTEGRATION

Since the measurement errors involved in estimating the threat position, the sensor position and the bearing angle arise from different measurement procedures, it is reasonable to assume that errors are independent. The probability densities required to perform the above calculations can be estimated as follows.

At time t_o an ocean surveillance sensor estimates the position of the threat to be located within an elliptical area characterized by the parameter set $E = \{X,Y,\theta,A,B\}$ with confidence $p_1 \times 100\%$. The elements of the parameter set E are respectively: X the latitude of the estimated threat position, Y the longitude of estimated threat position, θ the orientation of the major axis from true North, A the length of the semi-major axis, and B the length of the semi-minor axis. Since the measurement errors in determining threat position are generally assumed to be normally distributed, the ellipse characterized by E represents the minimal area $p_1 \times 100\%$ confidence region about the mean (X,Y). Treating this ellipse as the p_1 probability region of a bivariate normal distribution, a density function for the threat position at time t_o can be estimated. For convenience, locate the origin of a rectangular coordinate system at the center of the ellipse, (X,Y), with positive x-axis located along the major-axis of the ellipse at a bearing θ from true North. Assume a flat earth in the region of interest. Let $t_o = 0$. The mean of the threat position density $f_{X,Y}(x,y;0)$ is then the point (0,0). The variances in the X and Y directions can

be derived from the fact that the region with minimal area which contains the threat with probability p_1 is a k-sigma ellipse where k is determined from the relationship:

$$P[\text{threat located in k-sigma ellipse}] = 1 - e^{-k^2/2}$$

Thus,

$$\sigma_X{}^2 = (A/k)^2$$

and

$$\sigma_Y{}^2 = (B/k)^2$$

where

$$k = \sqrt{-2 \ln(1 - p_1)} \ .$$

X and Y are assumed to be independent.

If the course and speed of the threat are known with certainty to be ψ and s respectively, the probability density of the threat position at time late t_L can be shown to be again a bivariate normal with mean $(st_L\cos(\theta-\psi), st_L\sin(\theta-\psi))$ and variances $\sigma_X{}^2$, $\sigma_Y{}^2$. The p_1 probability region of the density at time t_L would then be an ellipse congruent to that characterized by the set E above but centered at the point $(st_L\cos(\theta-\psi), st_L\sin(\theta-\psi))$ (Figure 2) [Ref. 1].

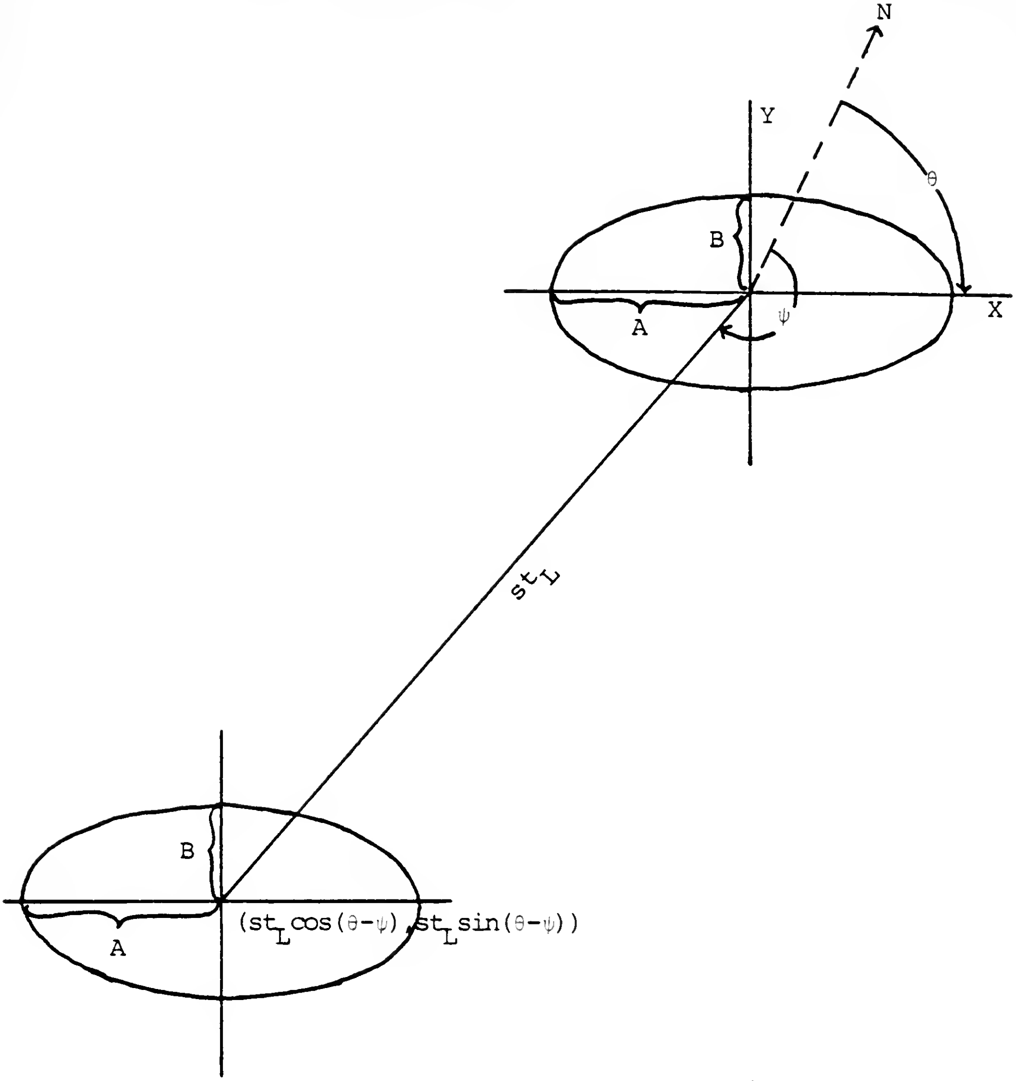


FIGURE 2. TIME LATE ELLIPSE, KNOWN COURSE AND SPEED

In some cases, the motion of a submarine on patrol in a large area can be characterized, when t_L is large, by an expansion of the probability area with time at a rate D. The result in such a case is that $f_{X,Y}(x,y;t_L)$ is still bivariate normal with mean (0,0) but with variances $\sigma_X^2 + Dt_L$ and $\sigma_Y^2 + Dt_L$.

When the motion of the threat cannot be described in the above manner, and the course and speed are not known, but assumed to be distributed according to the densities $f_\Psi(\psi)$ and $f_S(s)$, determining the probability density $f_{X,Y}(x,y;t_L)$ is a considerably more complex problem. Let $(x_o(s,\psi),y_o(s,\psi))$ be the coordinates of the point at which the threat would have to be located at time zero in order to reach the point (x_L,y_L) at time t_L if the threat speed were s and course ψ. Thus, $x_o(s,\psi) = x_L - st_L\cos(\theta-\psi)$ and $y_o(s,\psi) = y_L - st_L\sin(\theta-\psi)$ (Figure 3). Then,

$$f_{X,Y}(x,y;t_L) = \int_0^\infty \int_0^{360} f_{X,Y}[x_o(s,\psi),y_o(s,\psi);t_o] \cdot f_\Psi(\psi) f_S(s)\, d\psi ds \quad \text{[Ref. 1]}.$$

This density is no longer normal. In the special case when A = B, i.e., $\sigma_X^2 = \sigma_Y^2$, ψ has a uniform distribution over the interval (0°,360°) and S is known with certainty, as shown in Reference 2, the density changes with time as in Figure 4.

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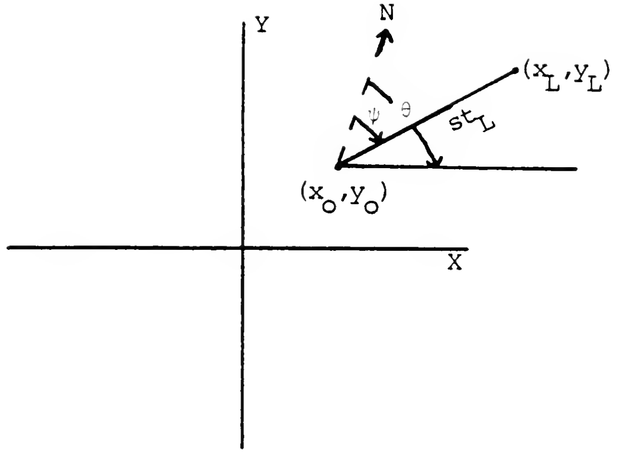

FIGURE 3. TIME LATE POSITION, COURSE AND SPEED UNCERTAIN

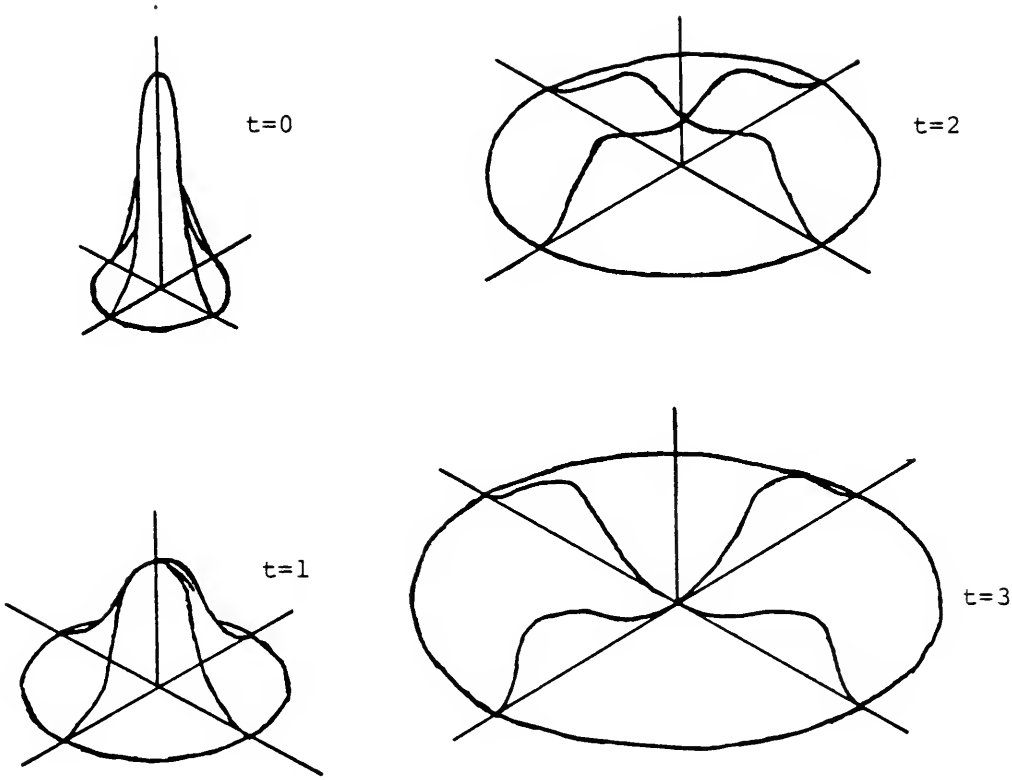

FIGURE 4. TIME LATE THREAT DENSITY, SPEED KNOWN COURSE UNIFORM (0°,360°)

The distribution of the bearing error measured by the local sensor can be estimated if the standard error of the sensor, σ_β, is known. The bearing error is then assumed to be normally distributed with mean zero and variance σ_β^2.

Assume that the estimated sensor location (u_o, v_o) is accurate to within R nautical miles with $p_3 \times 100\%$ confidence. The density $f_{U,V}(u,v)$ of sensor location can be assumed to be a circular bivariate normal with mean (u_o, v_o) and variance $\sigma^2 = R^2/k^2$. The value of k is determined from the relationship:

$$P[(u,v) \text{ contained in k-sigma region}] = 1 - e^{-k^2/2},$$

where the probability on the left is p_3 in this case.

Since the evaluation of $f'(\beta)$ considering sensor position density and bearing error density does not generalize to a closed form, algorithms are developed in the remainder of this paper for estimation of the probability (likelihood) that the threat identified by an external sensor lies on a line of bearing measured by a local sensor given that the threat is in region E, the sensor is in region C and the true line of bearing lies within (β_L, β_U).

III. ALGORITHMS

As indicated in the previous chapter, the variety of geometrical situations which can arise depending on the location of the sensor relative to the estimated threat location precluded development of a generalized analytical procedure. Rather, algorithms are developed in this paper for numerically evaluating $f'(\beta)$. The procedures applicable to a bivariate normal threat location density have been implemented on a TI-59 calculator. Appendix A contains a listing of that program. The calculations required when the time late threat location density is no longer normal exceeded the available program capacity of the TI-59 and therefore have not been implemented. A detailed verbal flow is provided at Appendix B for future implementation on a larger machine. The algorithms used in both situations are described in this chapter.

A. BIVARIATE NORMAL THREAT LOCATION DENSITY

This case includes situations (1) where the time elapsed since generation of the initial error ellipse by the ocean surveillance sensor is negligible, (2) where the motion of the threat can be assumed to be random in the manner described above, and (3) where course and speed of the threat are assumed to be known with certainty. With appropriate modifications to the input data, all three of these situations can be addressed using the program contained in Appendix A.

In situation (1) the data entered are the parameters of the ellipse as generated by the ocean surveillance sensor. In (2), the location and orientation from North of the ellipse is the same as originally generated, but the size of the ellipse expands at some constant rate of area per unit time which must be estimated by the user. This rate D, times the elapsed time, t_L, yields the factor which must be added to the semi-major and semi-minor axes of the original ellipse. That is, if the original error ellipse is a $p_1 \times 100\%$ confidence ellipse, the semi-major and semi-minor axes of the diffused ellipse will be input as A' and B' respectively:

$$A' = \sqrt{A^2 + (-2 \ln(1-p_1))Dt_L}$$

$$B' = \sqrt{B^2 + (-2 \ln(1-p_1))Dt_L}$$

In situation (3) the dimensions and orientation of the time late ellipse are the same as those of the original error ellipse, but the center of the ellipse is displaced from its original position by the known velocity times elapsed time. The updated position of the error ellipse is treated as the origin of the rectangular coordinate system for this situation and all linear measurements are made relative to this system. All angular measurements are made from true North.

Estimation of the likelihood that the threat lies on a bearing from the local sensor given that the threat is located

in a $p_1 \times 100\%$ confidence ellipse and the true bearing lies within the bounds (β_L, β_U) with confidence $p_2 \times 100\%$ proceeds as follows:

1. Estimate parameters of the bivariate normal density $f_{X,Y}(x,y)$ of threat location: $\mu_X = \mu_Y = 0$, σ_X^2, σ_Y^2.

2. Determine β_L and β_U such that an interval of length $2k_\beta \sigma_\beta$ centered on either β_L or β_U would contain β, the measured bearing, with probability p_2: $\beta_L = \beta - k_\beta \sigma_\beta$, $\beta_U = \beta + k_\beta \sigma_\beta$.

3. Determine sensor location coordinates relative to the origin of the threat ellipse.

4. Subdivide the angular interval (β_L, β_U) into 2n subintervals. Each subinterval k intersects the ellipse in a strip with average width W_k which corresponds to $\Delta\beta_t$.

5. At the midpoint of each subinterval k determine the equation of the line through the sensor position at the true bearing β_k from North.

6. Let the equation of the line of bearing β_k be $X = \frac{Y-c}{m}$. Then the plane perpendicular to the xy-plane which contains this line intersects the bivariate normal threat density in a curve whose equation is

$$g(y) = \frac{1}{2\pi\sigma_X\sigma_X} e^{-\frac{1}{2}\left(\frac{(y-c)^2}{m^2\sigma_X^2} + \frac{y^2}{\sigma_Y^2}\right)} \qquad (2)$$

found by making the substitution $X = \frac{Y-c}{m}$ in the density function $f_{X,Y}(x,y)$. It will prove convenient to expand the right side of (2) as follows [Ref. 3]:

$$g(y) = \frac{m}{\sqrt{2\pi}\sqrt{\sigma_Y^2 + m^2\sigma_X^2}} e^{-\frac{1}{2}\frac{c^2}{\sigma_Y^2 + m^2\sigma_X^2}} \tag{3}$$

$$\cdot \left[\frac{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}{\sqrt{2\pi}\,\sigma_X\sigma_Y m} e^{-\frac{1}{2}\left(\frac{\sigma_Y^2 + m^2\sigma_X^2}{m^2\sigma_X^2\sigma_Y^2}\right)\left(y - \frac{c\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}\right)^2}\right]$$

For computational purposes, assume that the width W_k of the region of the ellipse cut out by the angular subinterval around β_k will be nearly constant through the ellipse. Let the points of intersection of the line of bearing β_k with the threat ellipse be (X_{k1}, Y_{k1}) and (X_{k2}, Y_{k2}). Then approximate the volume of the normal density over this region by the absolute value of the product of the area under the curve g between Y_{k1} and Y_{k2} and W_k (Figure 5). Observe that the term in brackets in equation (3) above is the density function of a univariate normal random variable with mean

$$\frac{c\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}$$

and variance

$$\frac{m^2\sigma_X^2\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2} .$$

The term preceding the brackets in (3) is the slope of the line of bearing, m, times the density of a univariate normal random variable with mean zero and variance $\sigma_Y^2 + m^2\sigma_X^2$

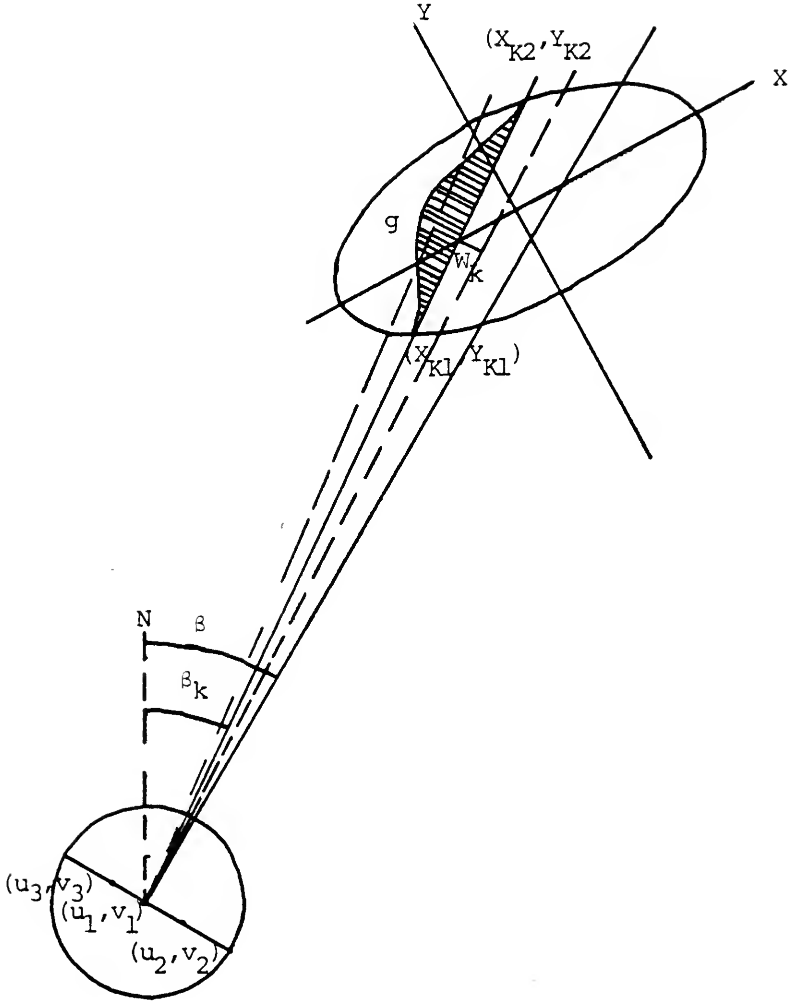

FIGURE 5

evaluated at c. Thus, evaluating the expression

$$\left| W_k \int_{Y_{k1}}^{Y_{k2}} g(y)\, dy \right|$$

is equivalent to the computation

$$\left| W_k m \left(\frac{1}{\sqrt{\sigma_Y^2 + m^2 \sigma_X^2}} \phi\left(\frac{c}{\sqrt{\sigma_Y^2 + m^2 \sigma_X^2}} \right) \right) (\Phi(Z_2) - \Phi(Z_1)) \right| ,$$

where

$$Z_i = \left(Y_{ki} - \frac{c\sigma_Y^2}{\sigma_Y^2 + m^2 \sigma_X^2} \right) \left(\frac{\sqrt{\sigma_Y^2 + m^2 \sigma_X^2}}{m\sigma_X \sigma_Y} \right)$$

for i = 1,2, Φ is the $P[Z \leq z]$ when Z is a standard normal random variable, and ϕ is the density function of a standard normal random variable.

If the substitution $Y = mX + c$ were made in the density $f_{X,Y}(x,y)$, an analogous situation would arise with the limits of the integration being X_{k1} and X_{k2}.

7. If the value of the density under the bivariate normal curve over the region of the ellipse defined by the subinterval $\Delta\beta_k$ is weighted by the probability of observing a bearing error $(\beta_k - \beta)$ the result is the probability of observing the bearing of the threat as β when the true threat location is in the segment of the ellipse defined by $\Delta\beta_k$. This probability is approximated as follows:

$$|W_k \int_{Y_{k1}}^{Y_{k2}} f_{X,Y}(\frac{y-c}{m},y)\,dy|\ f_\beta(\beta;\beta_k).$$

Since the bearing error is assumed normally distributed with mean zero and variance $\sigma_\beta^{\ 2}$, the value of $f_\beta(\beta;\beta_k)$ can be determined by the expression

$$\frac{1}{\sigma_\beta}\ \phi(\frac{\beta_k - \beta}{\sigma_\beta}),$$

where ϕ is as above the density function of a standard normal random variable.

8.

$$f'(\beta) = \int_{\text{all } \beta_t} f_\beta(\beta;\beta_t)\ f_{\beta_{TRUE}}(\beta_t)\ d\beta_t$$

is approximated by

$$\sum_{k=1}^{2n} |W_k[\int_{Y_{k1}}^{Y_{k2}} f_{X,Y}(\frac{y-c}{m},y)\,dy]|\ f_\beta(\beta;\beta_k)$$

The value of the sum is determined by repeating steps A.6 and A.7 above at the midpoints of each of the 2n subintervals defined in step A.4 and summing the results of each of these calculations. Obviously, the finer the subdivision of (β_L,β_U), the more accurate will be the estimate of the likelihood, but also the longer the calculation will take.

B. SENSOR POSITION UNCERTAINTY

The result of the above calculation will be the likelihood $f'(\beta)$ that the threat lies along the bearing measured by the sensor given that the threat is located within the threat ellipse and the true bearing is within the interval (β_L, β_U) and the sensor is at the position used to perform the calculation. We next will introduce additional calculations that are required to account for the fact that the sensor position is not known with certainty.

1. If the position of the sensor is estimated as being within R nautical miles of (u_o, v_o), its assumed coordinates in the xy-system, with $p_3 \times 100\%$ confidence, estimate the parameters of the sensor location density $f_{U,V}(u,v)$ with mean zero and variance σ^2: mean = (u_o, v_o), variance, $\sigma^2 = R^2/(-2 \ln(1 - p_3))$ in threat centered coordinates.

2. The bearing measured by the sensor is β regardless of the sensor location. Assume that the area of intersection of the angular wedge (β_L, β_U) and the threat ellipse does not change significantly as the sensor position is moved along the line of bearing β. Then the effect of the bivariate normal distribution of sensor location can be approximated by considering only the univariate normal density along a line through (u_o, v_o) perpendicular to β. Repeat the calculations in steps A.3 through A.8 above with the sensor located at each of the three points (u_o, v_o), $(u_o + .97\sigma\cos(\theta-\beta-90), v_o + .97\sigma\sin(\theta-\beta-90))$, and $(u_o - .97\sigma\cos(\theta-\beta-90), v_o - 97\sigma\sin(\theta-\beta-90))$. If the line through (u_o, v_o) perpendicular to β is subdivided

symmetrically about (u_o, v_o) such that 1/3 of the univariate normal density lies above each subinterval, the three points chosen above represent the "center of gravity" of each third of the density (Figure 6).

3. If R is chosen to include a significant proportion of the sensor density, i.e., on the order of 2σ or greater, the probability of the sensor being located in each of the three regions is approximately 1/3. Thus, if p_3 is on the order of .86, multiply each result in step B.2 by 1/3.

4. Summing the results of steps B.2 and B.3 yields an estimate of the likelihood that the threat lies at bearing β given the threat is in the $p_1 \times 100\%$ confidence ellipse, β_T is in (β_L, β_U) and the sensor is in a $p_3 \times 100\%$ confidence region. That is, the likelihood is estimated by

$$f'(\beta) = \sum_{j=1}^{3} \frac{1}{3} f_j'(\beta) ,$$

where j is the index of sensor position in figures 5 and 6.

Instructions for application of the TI-59 program to calculate the above are included in Appendix A.

C. THREAT DISTRIBUTION NOT BIVARIATE NORMAL AFTER TIME LATE ELAPSED

The basic approach to evaluating $f'(\beta)$ when the time late distribution of the threat is no longer bivariate normal is the same as that just discussed. The principal difference

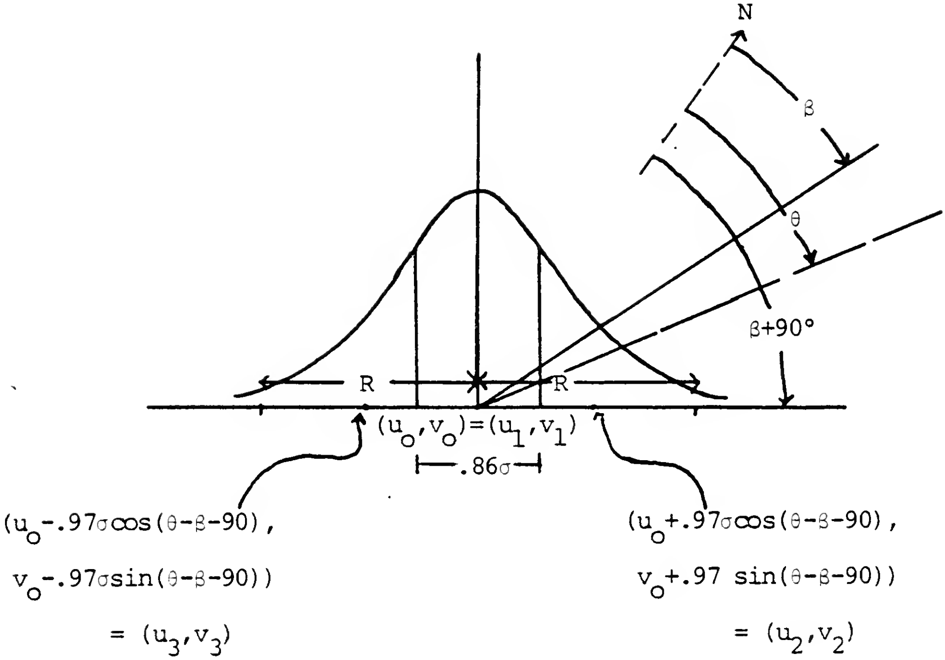

FIGURE 6. ESTIMATE OF SENSOR POSITION DENSITY

arises from the fact that the time late density is significantly more complex in this case.

The threat distribution becomes distorted from the normal after some time late when the course and speed of the threat are constant but not known with certainty. Application of the method described herein requires that the user assume a discrete distribution of the speed of the threat with upper and lower bounds S_M and S_l, respectively. In addition, the threat course is assumed to be uniformly between 0° and 360°.

The density of threat location after some time late t_L when the threat speed is s_i then becomes [Ref. 1]:

$$f_{X,Y}(x,y;t_L,s_i) = \frac{1}{2\pi\sigma_X\sigma_Y}\int_0^{360} \exp\left[-\frac{1}{2}\left(\frac{(x-s_i t_L\cos(\theta-\psi))^2}{\sigma_X^2} + \frac{(y-s_i t_L\sin(\theta-\psi))^2}{\sigma_Y^2}\right)\right]\frac{d\psi}{360}$$

where θ is the bearing of the major axis from North. Note that the new threat density is still centered at the same position as the time zero ellipse but its shape changes as in Figure 4 of Chapter II. If the threat speed is s_i, and the course is uniformly distributed over (0°,360°), the outer limit of the new planar region containing the threat after time t_L has elapsed, given that it was originally located in the $p_1 \times 100\%$ confidence ellipse with semi-axes A and B, can be represented by an ellipse with semi-major axis $A+s_i t_L$ and semi-minor axis $B+s_i t_L$ (Figure 7). Thus the

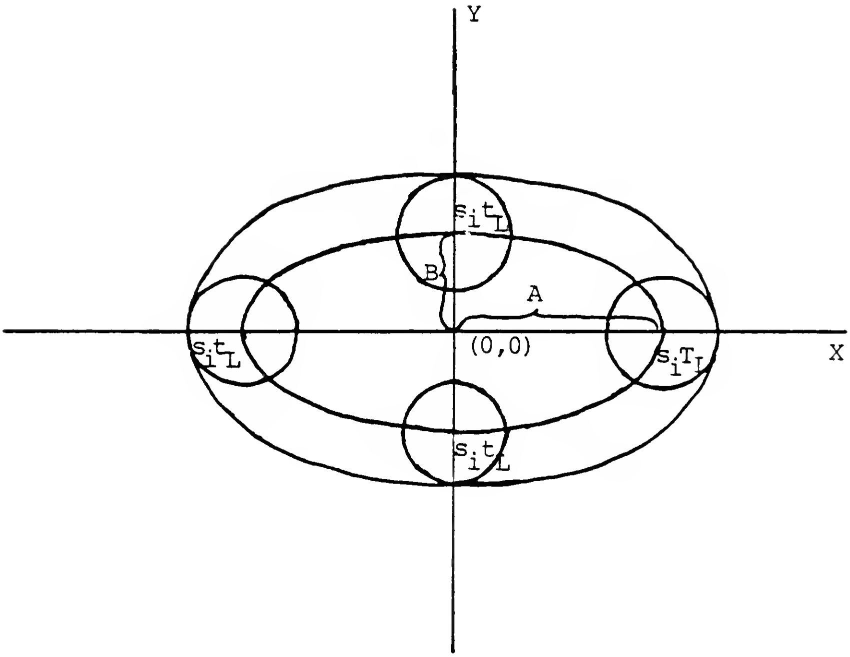

FIGURE 7. TIME LATE PLANAR REGION

region over which the density will be evaluated is still the intersection of an ellipse with an angular wedge, recalling however that the density is no longer normal. Further, the new elliptical region does not represent a $p_1 \times 100\%$ confidence region of the time late density.

Calculation of the likelihood that the threat lies along bearing β given that the threat lies in the $p_1 \times 100\%$ ellipse at time zero, that the true bearing lies in (β_L, β_U), that the sensor lies in the $p_3 \times 100\%$ circle and that the speed is s_i proceeds as follows:

1. Estimate the parameters of the original normal distribution: mean = (0,0), $\sigma_X{}^2 = A^2/(-2 \ln(1-p_1))$, $\sigma_Y{}^2 = B^2/(-2 \ln(1-p_1))$.

2. Determine the upper and lower bounds on the true bearing wedge, $\beta + k_\beta \sigma_\beta$ and $\beta - k_\beta \sigma_\beta$.

3. Determine the time late planar region as the ellipse with semi-major axis equal to $A + s_i t_L$ and semi-minor axis equal to $B + s_i t_L$.

4. Determine the position of the sensor relative to the ellipse center.

5. Subdivide the bearing fan $(\beta - k_\beta \sigma_\beta, \beta + k_\beta \sigma_\beta)$ into 2n subintervals.

6. At the midpoint of each subinterval, determine the equation of the line through the ship position at that bearing β_k.

7. Let the equation of the line of bearing β_k be $X = \frac{Y-c}{m}$. Then, the plane perpendicular to the xy-plane which

contains this line intersects the time late threat density in a curve whose equation is

$$g_L(y) = \left(\frac{m}{\sqrt{2\pi}\sqrt{\sigma_Y^2 + m^2\sigma_X^2}} e^{-\frac{1}{2}\frac{c^2}{\sigma_Y^2 + m^2\sigma_X^2}}\right)\left(\frac{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}{\sqrt{2\pi}\, m\, \sigma_X\, \sigma_Y}\right.$$

$$\cdot\ e^{-\frac{1}{2}\left[\left(\frac{\sigma_Y^2 + m^2\sigma_X^2}{m^2\sigma_X^2\sigma_Y^2}\right)\left(y - \frac{c\,\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}\right)^2\right]}$$

$$\cdot \int_0^{360} \exp\left[-\frac{1}{2}\left(\frac{(s_it_L)^2\cos^2(\theta-\psi) - 2s_it_Ly\cos(\theta-\psi)}{m^2\sigma_X^2}\right.\right.$$

$$\left.\left. + \frac{(s_it_L)^2\sin^2(\theta-\psi) - 2s_it_Ly\sin(\theta-\psi)}{\sigma_Y^2}\right)\right] \frac{d\psi}{360} \qquad (4)$$

found by making the substitution $x = \frac{y-c}{m}$ in the density function $f_{X,Y}(x,y;t_L)$ and expanding.

For computational purposes, assume that the width W_k of the region of the time late threat ellipse cut out by the angular subinterval around β_k will be nearly constant through the ellipse. Let the points of intersection of the line of bearing β_k with the time late threat ellipse be (X_{k1}, Y_{k1}), (X_{k2}, Y_{k2}). Then approximate the time late density at speed s_i over this region by the absolute value of the product of W_k and the area under the curve g_L between Y_{k1} and Y_{k2}. The

area under curve g_L can be approximated as follows. Subdivide the interval (Y_{k1}, Y_{k2}) into n_1 segments of length h. Evaluate g_L at the midpoint of each segment, y_J. Note that the first term in parenthesis in (4) is a constant equal to m times the value of the density of a normal random variable with mean zero and variance $\sigma_Y^2 + m^2\sigma_X^2$ evaluated at c. The second term in parenthesis is the density function of a normal random variable with mean

$$\frac{c\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}$$

and variance

$$\frac{m^2\sigma_X^2\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}$$

evaluated at y. The variable y also appears in the integral term in equation (4). Numerically evaluate this term of (4) with $y = y_J$. Let $g_I(y_J)$ be the result of this computation. Then, evaluating the area under the curve g_L between Y_{k1} and Y_{k2} is equivalent to the calculation:

$$\sum_{J=1}^{n_1} \frac{m}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}} \phi\left(\frac{c}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}\right) \frac{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}{m\sigma_X\sigma_Y} \phi(z_J)\ g_I(y_J)$$

where $\phi(\cdot)$ is the density function of a standard normal random variable, and

$$z_J = \left(y_J - \frac{c\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}\right)\left(\frac{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}{m\sigma_X\sigma_Y}\right) .$$

An analagous situation arises if the substitution $Y = mX + c$ is made for y in the time late density $f_{X,Y}(x,y;t_L)$.

8. The value of the density over the subinterval containing β_k is then weighted by the instantaneous probability that the bearing error is $\beta_k - \beta$:

$$\left| W_k \int_{Y_{k1}}^{Y_{k2}} f_{X,Y}(x_L,y_L;t_L)\,dy_L \right| f_\beta(\beta;\beta_k)$$

9. Steps C.7 and C.8 are then repeated for each subinterval of (β_L,β_U) and the results of each calculation are summed.

10. The uncertainty in sensor location is accounted for by repeating steps C.4 through C.9 with the assumption the sensor is located at each of the three points in Figure 5, multiplying by the probability the sensor lies in that interval and summing each result.

11. The result of calculations in steps C.1 through C.10 is the likelihood that the threat lies on bearing β given the threat was originally located in the $p_1 \times 100\%$ confidence ellipse, $\beta_T \in (\beta_L,\beta_U)$, the sensor is located in the $p_3 \times 100\%$ circle and the threat speed is s_i. The condition that the speed is s_i is removed by repeating the

calculations C.1 through C.10 for each of the speeds s_i, $i = 1,\ldots,M$ multiplying the result by the probability that the speed equals s_i and summing all M results. The final result is the likelihood that the threat lies on β given it was originally located in the $p_1 \times 100\%$ ellipse, $\dot{\beta}_T \in (\beta_L, \beta_U)$, and the sensor is in the $p_3 \times 100\%$ circle estimated as

$$\sum_{i=1}^{M} P[S = s_i] \sum_{j=1}^{3} \frac{1}{3} f_j'(\beta) \ .$$

IV. CONCLUSIONS

Possible applications of and extensions to the algorithms developed in Chapter III are discussed in this chapter.

As indicated in Chapter I, the objective of this paper has been to develop a means of assessing the likelihood a threat whose position has previously been estimated lies at a given measured bearing from a local sensor. The procedures were developed with a view towards permitting the user to make comparisons among lines of bearing measured by different sensors or among conflicting bearing information generated by one sensor. The approach chosen has been to estimate the likelihood that a threat lies on bearing β given that the threat is located in an ellipse of specified confidence $p_1 \times 100\%$, that β is measured with $p_2 \times 100\%$ accuracy, and that the sensor position is measured with $p_3 \times 100\%$ accuracy. The algorithm to calculate the likelihood in the cases where the probability distribution of the target can be assumed to be bivariate normal at the time of the bearing measurement has been implemented on the TI-59 calculator. The cases in which this program applies are the following: (1) when the time elapsed since generation of the threat error ellipse is small enough to justify using the original estimate of the ellipse; (2) when threat course and speed are known, in which case the ellipse center is translated from the original position according to the course, speed and elapsed time information;

(3) when the threat can be assumed to be moving about in a random manner over a significant region in such a way that the ellipse center remains unchanged, but the x and y variances have increased. When none of these cases hold, but the course is assumed uniformly distributed over (0°,360°) and speed has a discrete distribution over a finite interval, the threat density at the time late t_L is not a bivariate normal density. The algorithm applicable in this case has not been implemented, but is described in some detail in Chapter III and Appendix B.

Once the appropriate computation has been completed for each of the bearings considered, the results can be used to weight the value of several bearings in refining the threat location estimate provided by the external sensor. Note that, although unlikely, $f'(\beta)$ may correctly be greater than one. Comparisons using these likelihoods should be made only when the upper and lower bounds on the true bearing fan for each bearing are chosen at the same probability level p_2 and the uncertainty areas for all sensors include the same probability level p_3. Further, these likelihood levels should be selected so as not to exclude a significant portion of the appropriate density. If p_2 or p_3 are not the same in all cases to be compared, $f'(\beta)$ must be divided by the applicable value of p_2 or p_3 for each measured bearing β to be considered.

Having established the relative value of available bearing information, the user can allocate weapons or further search effort accordingly. However, the probabilities calculated

are strictly ordinal data and do not define a redistribution of target location probability based on additional information. Further, the threat ellipse does not contain the target with certainty. The power to predict the probability of success of a search or weapons allocation plan based on the priorities established by these procedures is limited by these constraints. In this area in particular further research would be useful.

In situations such as that for which the procedures in this paper have been developed, where a track has not been developed on the target, introduction of unrelated bearing data to a target motion model could impact significantly on the reliability of future position predictions. If there is high confidence in the reliability of the estimate of the threat ellipse provided by the ocean surveillance sensor, the prioritization established herein could be used to process bearing data prior to input to a target motion model. Using a pre-established threshold, only those bearings which coincide with the threat with an acceptable level of likelihood could be used to refine or update a track on the threat.

Desirable enhancements to the algorithms include providing for the instances in which the interval of uncertainty of the target course is known to be less than $(0°,360°)$. Further, if the circular region of radius R contains the sensor with significantly less than 86% confidence or the assumption of a

bivariate normal distribution of sensor location is unsatisfactory, it is left to the user to modify the calculations accordingly.

The utility of these algorithms would also be improved by implementation on a larger and faster system than the TI-59 calculator.

APPENDIX A. TI-59 PROGRAM VERBAL FLOW AND USER'S INSTRUCTIONS

Part I Step Number	Verbal Flow
000 - 029	Enter the confidence level p_1 for the threat ellipse. Calculate the value of k for the given p_1: $k = \sqrt{-2 \ln(1-p_1)}$.
020 - 029	Enter the length of the semi-major axis, A. Calculate $\sigma_X = A/k$.
030 - 052	Enter the length of the semi-major axis, B. Calculate $\sigma_Y = B/k$. Calculate B/A, $\sigma_X\sigma_Y$, σ_X^2 and σ_Y^2.
053 - 058	Enter orientation of semi-major axis, θ.
059-063	Enter bearing from sensor to center of threat ellipse, α. Calculate $\theta-\alpha$.
064 - 073	Enter distance r from sensor position to center of threat ellipse. Determine rectangular coordinates of sensor position (U,V) from polar coordinates $(-r,\theta-\alpha)$.
074 - 089	Store the constants 360, $\sqrt{2\pi}$. Initialize register 35 to 0.

090 - 096	Enter number of standard deviations desired for bearing fan, k_β.
097 - 104	Enter standard deviation of bearing error, σ_β. Calculate $k_\beta\sigma_\beta$.
105 - 143	Enter the angular stepsize desired, $\Delta\beta$, for incrementally stepping through (β_L, β_U). Calculate $\frac{1}{2}\Delta\beta$. Calculate the largest number n of increments of size $\Delta\beta$ contained in $k_\beta\sigma_\beta - \frac{1}{2}\Delta\beta$ degrees. Initialize counter 00 to n+1. Save n+1 in register 20. Determine $$\delta\Delta\beta = k_\beta\sigma_\beta - \tfrac{1}{2}\Delta\beta - n\Delta\beta ,$$ the residual increment. Calculate $\frac{1}{2}\delta\Delta\beta$.
144 - 146	Initialize counter 01 to 2. Calculations will be made at the midpoint of each interval from $\beta + \frac{1}{2}\Delta\beta$ to $\beta + k_\beta\sigma_\beta$, then at the midpoint of each interval from $\beta - \frac{1}{2}\Delta\beta$ to $\beta - k_\beta\sigma_\beta$, and finally at β. Counter 01 indicates whether calculations are complete on both sides of β.
147 - 151	Enter bearing measured by sensor, β.
152 - 154	Enter standard deviation of sensor position σ.

155	Enter index of the sensor position to be used for this run.
156 - 157	Coordinates of the sensor position are selected in accordance with run number entered above in Subroutine sin.
158 - 238	Determine whether a bearing β' parallel to either axis is included in the fan (β_L, β_U). If (β_L, β_U) includes a bearing parallel to the y-axis, use program 1 for Part II. The appropriate program number is displayed in calculator display register. If (β_L, β_U) does not contain a bearing parallel to either axis, use program 1.
239 - 259	Subroutine P/R.
260 - 340	Subroutine sin.

index of the [illegible]

to be used for this run.

Part II, Programs 1 and 2

Step Number	Verbal Flow
000 - 003	Initiate β'.
004 - 007	If the last angular increment on this side of β has been considered, go to step 513. Otherwise continue.
008 - 012	Decrement counter for angular increment. If counter = 0, go to 021. Otherwise continue.
013 - 020	Remove flag to indicate this is not the last angular increment. Recall $\Delta\beta$, the input angular step-size. Go to 029.
021 - 028	Set flag to indicate this is the last angular increment on this side. Add one-half the input angular stepsize $\Delta\beta$ and one-half the residual stepsize $\delta\Delta\beta$: $\frac{1}{2}\Delta\beta + \frac{1}{2}\delta\Delta\beta$.
029 - 030	Increment β' by the appropriate stepsize.
031 - 045	Convert β' to an angle between 0° and 360°.
046 - 051	Calculate $\theta-\beta'$. Print $\theta-\beta'$.

052 - 064	If $\lvert\theta-\beta'\rvert = 90°$ or $= 270°$, go to 236.
065 - 074	If $\lvert\theta-\beta'\rvert = 0°$ or $= 180°$, go to 075. Otherwise go to 133.
075 - 086	If the absolute value of the y coordinate of sensor position is greater than the length of the semi-minor axis, β, go to 004. In this case β' does not intersect the error ellipse. Otherwise continue.
087 - 088	Set flag 2 to indicate that the bearing β' is parallel to the x-axis.
089 - 111	Calculate the coordinates of the points of intersection of β' with the threat ellipse. The y-coordinates are equal to V, the y-coordinate of sensor position. x-coordinates are determined in Subroutine y^x. The points of intersection are symmetric about the y-axis. Thus, $X_{k1} = -X_{k2}$. Store the smaller x value in register 27, the larger in register 28.
112 - 132	Save locations of X_{k1}, X_{k2}, Y_{k1}, σ_X, and σ_Y. Go to 291.
133 - 138	Remove flags 2 and 3 to indicate that β' is not parallel to either the x- or y- axes.

139 - 145 Calculate the slope of β':

$$m = \tan(\theta-\beta').$$

146 - 154 Calculate the y-intercept of β':

$$c = V - mU.$$

155 - 176 If $c^2 > A^2m^2 + B^2$, go to 004. In this case, β' does not intersect the error ellipse. Otherwise continue.

177 - 235 Calculate X_{k1} and X_{k2}, the x-coordinates of the points of intersection of β' with the ellipse:

$$X_{k1} = \frac{-mc + \frac{B}{A}\sqrt{A^2m^2 + B^2 - c^2}}{\frac{A^2m^2 + B^2}{A^2}}$$

$$X_{k2} = \frac{-mc - \frac{B}{A}\sqrt{A^2m^2 + B^2 - c^2}}{\frac{A^2m^2 + B^2}{A^2}} .$$

Calculate the y-coordinates of the points of intersection of β' with the ellipse, Y_{k1}, Y_{k2}:

$$Y_{k1} = mX_{k1} + c$$

$$Y_{k2} = mX_{k2} + c .$$

Go to 299.

236 - 247 If $|U|$, the absolute value of the x-coordinate of sensor position, is greater than the length of the semi-major axis, A, go to 004. β'

does not intersect the error ellipse in this case. Otherwise continue.

248 - 249 Set flag 3 to indicate that β' is parallel to the y-axis.

250 - 272 Calculate the coordinates of the points of intersection of β' with the error ellipse. The x-coordinates are equal to U, the x-coordinate of sensor position. The y-coordinates are determined in Subroutine y^x. The points of intersection are symmetric about the x-axis. Thus, $Y_{k1} = -Y_{k2}$. Store the smaller y value in register 29, the larger in 30.

273 - 298 Save the locations of Y_{k2}, Y_{k1}, X_{k1}, σ_Y, σ_X.

299 - 328 Determine the width of the strip around β', W:

If flag 0 set,

$$W = (d_1+d_2) \tan(\tfrac{1}{2}\delta\Delta B),$$

Otherwise

$$W = (d_1+d_2) \tan \tfrac{1}{2}\Delta B).$$

d_1 and d_2 are the distances from (U,V), the sensor position, to the intersection points (X_{k1},Y_{k1}) and (X_{k2},Y_{k2}), respectively. Distances are calculated in Subroutine log.

Multiply W by $\Delta\beta$ if flag 0 not set. Otherwise, multiply by $\delta\Delta\beta$. Save result in register 37.

329 - 334 If β' is parallel to either the x- or the y-axis, go to 468. Otherwise continue.

<u>When using program 1</u>:

335 - 380 Express the y-coordinates of the intersection points as standard normal random variables, Z_1 and Z_2:

$$Z_i = \frac{Y_{ki} - \dfrac{c\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}}{\dfrac{|m|\ \sigma_X\sigma_Y}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}},$$

$i = 1,2.$

381 - 391 Sort the values Z_i in descending order. Let Z_2' be the larger value. Z_1' is the smaller.

392 - 401 Calculate $\Phi(Z_2') - \Phi(Z_1')$, the probability that a standard normal random variate lies between Z_1' and Z_2'.

402 - 424 Multiply the results of steps 392-401 by:

$$\frac{1}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}} \quad \phi\left(\frac{c}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}\right) \times |m| \times W,$$

where ϕ is the standard normal density function.

The result of the calculations in steps 335-424 is

$$\left| W \int_{Y_{k1}}^{Y_{k2}} f_{X,Y}\left(\frac{y-c}{m}, y\right) dy \right| .$$

<u>When using program 2:</u>

335 - 372 Express the x-coordinates of the intersection points as standard normal random variables Z_1 and Z_2:

$$Z_i = \frac{X_{k2} + \dfrac{mc\sigma_X^2}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}}{\dfrac{\sigma_X\sigma_Y}{\sqrt{\sigma_X^2 + m^2\sigma_Y^2}}} ,$$

$i = 1,2$.

373 - 383 Sort the values of Z_i in descending order. Let Z_2' = larger value. Z_1' = smaller.

384 - 393 Calculate $\Phi(Z_2') - \Phi(Z_1')$, the probability that a standard normal random variate lies between Z_1' and Z_2'.

394 - 413 Multiply the results of steps 384-393 by:

$$\frac{1}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}} \phi\left(\frac{c}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}\right) \times W,$$

where ϕ is the standard normal density function.

The result of the calculations in steps 335-413 is

$$\left| W \int_{Y_{k1}}^{Y_{k2}} f_{X,Y}(x, mx+c)\ dx \right| .$$

Regardless of which program is in use:

425 - 465 Multiply results of previous calculation by

$$\frac{1}{\sigma_\beta} \phi\left(\frac{\beta''-\beta}{\sigma_\beta}\right)$$

where $\beta'' = \begin{cases} \beta' & , \beta' < 180° \\ \beta'-360° & , \beta' \geq 180° \end{cases}$

and ϕ is the standard normal density.

The result of this calculation is

$$f_\beta(\beta;\beta') \left| W \int f_{X,Y}(x,y)\ dy \right| .$$

466 - 467 Go to 508

468 - 507 Calculate the area under the curve formed by the intersection of β' with $f_{X,Y}(x,y)$. When β' parallels the x-axis this calculation becomes:

$$\frac{1}{\sigma_Y} \phi(\frac{V}{\sigma_Y}) [\Phi(Z_2) - \Phi(Z_1)] ,$$

where $Z_2 = X_{k2}/\sigma_X$ and $Z_1 = X_{k1}/\sigma_X$, V is the y-coordinate of the sensor position.

When β' is parallel to the y-axis, the calculation is:

$$\frac{1}{\sigma_X} \phi(\frac{U}{\sigma_X}) [\Phi(Z_2) - \Phi(Z_1)] ,$$

where $Z_2 = Y_{k2}/\sigma_Y$ and $Z_1 = Y_{k1}/\sigma_Y$, U is the x-coordinate of the sensor position.

Go to 422 to complete calculation of $f_\beta(\beta;\beta') | W \int f_{X,Y}(x,y) |$.

(Go to 410 in Program 2).

508-512 Accumulate the probability at each angular interval. Display result.

513 - 515 If flag 1 is set go to 558. Otherwise continue.

516 - 520 If flag 0 is not set, that is if the angular increment just considered was not the last on this side of β, go to 004 and continue calculation on same side of β. Otherwise continue.

521 - 524 Decrement the counter 01. If counter is now 0 go to 543. In this case, the probabilities have been calculated on both sides of β. The calculation at β remains to be done. Otherwise continue.

525 - 542 Remove flag 0. Multiply $\Delta\beta$, $\frac{1}{2}\Delta\beta$, and $\frac{1}{2}\delta\Delta\beta$ by -1. Reinitialize counter 00 to n+1. Go to 000 to begin calculation on second side of β.

543 - 567 Set flag 01 indicating that calculations on both sides of β have been completed. The next iteration will do the calculation at $\beta' = \beta$. Remove flag 00. Initialize β' to 0. Recall β. Go to 029.

558 - 566 Remove flag 01. Display the accumulated likelihood. STOP

The result of this calculation is approximately $f_j'(\beta)$ with sensor at (U_j, V_j) where j = run number entered in Part I.

567 - 587 Subroutine y^x calculates the points of intersection when β' is parallel to either the x or y axis:

$$Y_{k1} = Y_{k2} = V$$

$$X_{k1} = -A\sqrt{1 - (\frac{U}{B})^2}$$

$$X_{k2} = A\sqrt{1 - (\frac{U}{B})^2}$$

Parallel to y-axis:

$$X_{k1} = X_{k2} = U$$

$$Y_{k1} = -B \sqrt{1 - (\frac{V}{A})^2}$$

$$Y_{k2} = B \sqrt{1 - (\frac{V}{A})^2} \quad .$$

589 - 610 Subroutine log calculates the distance from sensor position to point of intersection of β' with the ellipse:

$$d_i = \sqrt{(U-X_{ki})^2 + (V-Y_{ki})^2}$$

Part III Step Number	Verbal Flow
000 - 012	Enter result of run 1. Multiply by 1/3.
013 - 021	Enter result of run 2. Multiply by 1/3.
022 - 029	Enter result of run 3. Multiply by 1/3.
030 - 032	Display likelihood.
033 - 041	If p_2 is the same for all bearings to be compared, enter 1. Go to 042. Otherwise, enter p_2 for this bearing. Divide likelihood by p_2. Display result.
042 - 050	If p_3 is the same for all bearings to be compared, enter 1 and STOP. Otherwise, enter p_3 for this bearing. Divide likelihood by p_3. Display result. STOP.

USER'S INSTRUCTIONS

The program to determine the likelihood that the threat lies along bearing β given the threat is in the confidence ellipse, $\beta_T \varepsilon (\beta_L, \beta_U)$ and the sensor is within the $p_3 \times 100\%$ confidence circle is in three parts. All parts require the use of a printer and the use of the Applied Statistics Library Module. Prior to running, the calculator must be repartitioned:

1. Enter 4

2. Press 2nd OP 17

Part I

1. Read sides 1 and 2 of Part I

2. Read side 4 of Part II either program 1 or program 2. Since program 2 is used more often, unless it is known that the bearing fan (β_L, β_U) contains a bearing parallel to the major axis, recommend using program 2 of Part II.

3. Press RST

4. Enter p_1, the confidence level of the threat ellipse.
Press A

5. Enter A, length of the semi-major axis. If data provided is length of the entire major axis, divide by 2 before entering.
Press B.

6. Enter B, length of the semi-minor axis. If data provided is length of entire minor axis, divide by 2 before entering. Press C.

7. Enter θ, the bearing of major axis from North. $0° \leq \theta \leq 180°$. θ is entered in degrees. Press D.

8. Enter α, the bearing of the threat ellipse center from the estimated sensor position. $0° \leq \alpha \leq 360°$, in degrees. Press E.

9. Enter r, the range from the estimated sensor position to the center of the threat ellipse. Press 2nd A'.

10. Enter k_β, the number of standard deviations desired in one direction from β. The program will construct a fan of equal size on the other side of β. Press 2nd B'.

11. Enter σ_β, the standard deviation of bearing error. Press 2nd C'.

12. Enter $\Delta\beta$, the desired angular stepsize. Press 2nd D'.

13. Enter β, the measured bearing. Press 2nd E'.

14. Enter σ, the standard deviation of sensor position. Press R/S.

15. Enter the number of this run, 1, 2 or 3. When run number = 1, sensor is located at its estimated position (u_o,v_o). When run number = 2, the location will be $(u_o + .97\sigma\cos(\theta-\beta-90), v_o + .97\sigma\sin(\theta-\beta-90))$. When run number = 3, the location will be $(u_o - .97\sigma\cos(\theta-\beta-90), v_o - .97\sigma\sin(\theta-\beta-90))$. Press R/S. If program number displayed matches that of the Part II side 4 read in, continue to 16. Otherwise, press 2nd CMS, RST. Read in side 4 of the Part II program which corresponds to the number displayed. Repeat 3 through 15.

16. Press 4 2nd WRITE. Rerecord side 4 of the Part II program read in. This enables data entered in Part I to be transferred to Part II.

Part II

1. Read sides 1, 2, 3 and 4 of Part II Program 1 or 2 as selected by the Part I program.

2. Press RST

3. Press R/S. Values of β_t and $W\int f_\beta(\beta;\beta_t) f_{\beta_{TRUE}}(\beta_t)$ will print alternately. Final result also prints out at end of calculation.

4. Record final result: Likelihood threat lies along bearing β given threat is in $p_1 \times 100\%$ ellipse, true bearing is in $p_2 \times 100\%$ fan and sensor is at location used in this run. Parts I and II must be completed 3 times (Run numbers 1, 2 and 3) before proceeding to Part III if uncertainty in sensor position is being considered.

Part III

1. Read side 1 of Part III.

2. Enter result of run 1. Press A.

3. Enter result of run 2. Press B.

4. Enter result of run 3. Press C.

5. Enter p_2, confidence level of (β_L, β_U), if necessary. Otherwise enter 1. Press D.

6. Enter p_3 confidence level of sensor position, if necessary. Otherwise, enter 1. Press E.

APPENDIX B. TIME LATE VERBAL FLOW

VERBAL FLOW

1. Enter p_1. Determine the value of k: $k = \sqrt{-2 \ln (1 - p_1)}$

2. Enter A. Determine $\sigma_X^2 = (A/k)^2$

3. Enter B. Determine $\sigma_Y^2 = (B/k)^2$

4. Enter θ, orientation of major axis

5. Enter α, bearing from sensor to ellipse center

6. Enter range r from sensor to ellipse center

7. Calculate (u_o, v_o) the coordinates of mean sensor position. $u_o = -r \cos(\theta-\alpha)$, $v_o = -r \sin(\theta-\alpha)$

8. Enter σ_β, the standard deviation of bearing error

9. Enter k_β, the number of standard deviations to be included in the bearing fan on each side of β.

10. Enter desired angular stepsize, $\Delta\beta$

11. Calculate the number of iterations of size $\Delta\beta$ required on each side of β: $I = [(k_\beta \sigma_\beta - \frac{1}{2}\Delta\beta)/\Delta\beta]$ where $[\cdot]$ means the greatest integer less than or equal to the value within.

12. In general $(k_\beta \sigma_\beta - \frac{1}{2}\Delta\beta)/\Delta\beta$ is not an integer. Determine the size of the fractional increment:

$$\delta\Delta\beta = ((k_\beta \sigma_\beta - \frac{1}{2}\Delta\beta)/\Delta\beta - I)\Delta\beta.$$

13. Enter β, the measured bearing. Let PROB2=0, PROB1=0

14. Enter the time late t_L

15. For each of the discrete threat speeds to be considered, repeat steps 16 to 43. Then go to 44.

16. Enter the target speed s. Let PROB = 0

17. Expand the outer limit of the threat ellipse:

$$\text{Let} \quad A' = A + st_L$$
$$B' = B + st_L$$

18. Let $\beta' = \beta$

19. For each of the I increments of size $\Delta\beta$ repeat steps 20 to 38.

20. Let $\beta' = \beta' + \Delta\beta$

21. Calculate $(\theta-\beta')$. If $|\theta-\beta'| = 0°$ or $180°$, go to 27. If $|\theta-\beta'| = 90°$ or $270°$, go to 30

22. Calculate $m = \tan(\theta-\beta')$, the slope of the line of bearing.

23. Calculate $c = mU+V$, where (U,V) is the sensor location for this iteration. c is the y-intercept of the line of bearing.

24. Calculate $m^2A^2 + B^2 - c^2$. If this quantity is less than zero, the line of bearing β' does not intersect the threat ellipse. Go to step 20 and process next increment of size $\Delta\beta$, if any remain. If all I intervals of size $\Delta\beta$ on this side of β have been considered go to step 40.

25. Calculate the points of intersection of the line of bearing with the expanded ellipse:

$$X_1 = \frac{(-mc + (B'/A')\sqrt{A'^2m^2 + B'^2 - c^2})(A'^2)}{A'^2m^2 + B'^2}$$

$$X_2 = \frac{(-mc - (B'/A')\sqrt{A'^2m^2 + B'^2 - c^2})(A'^2)}{A'^2m^2 + B'^2}$$

$$Y_1 = mX_1 + c, \quad Y_2 = mX_2 + c$$

26. Go to 32

27. If $|V| > B'$, the bearing β' parallel to the x-axis does not intersect the threat ellipse. Go to step 20 and process next increment of size $\Delta\beta$ if any remain. If all I intervals if size $\Delta\beta$ on this side of β have been considered, go to 40.

28. Calculate the points of intersection

$$Y_1 = Y_2 = V$$

$$X_1 = -B'\sqrt{1 - (V^2/B'^2)}$$

$$X_2 = B'\sqrt{1 - (V^2/B'^2)}$$

29. Go to 32

30. If $|U| > A'$, the bearing β' parallel to the y-axis does not intersect the threat ellipse. Go to step 20 and process the next increment of size $\Delta\beta$ if any remain. If all I intervals of size $\Delta\beta$ on this side of β have been considered, go to 40.

31. Calculate the points of intersection:

$$X_1 = X_2 = U$$

$$Y_1 = -A'\sqrt{1 - (U^2/A'^2)}$$

$$Y_2 = A'\sqrt{1 - (U^2/A'^2)}$$

32. Calculate the median width of the strip of the ellipse defined by the angular subinterval under consideration. If the subinterval is of size $\Delta\beta$:

$$W = \frac{d_1 + d_2}{2}(2 \tan \tfrac{1}{2}\Delta\beta)$$

If the subinterval is of size $\delta\Delta\beta$:

$$W = \frac{d_1 + d_2}{2}(2 \tan \tfrac{1}{2}\delta\Delta\beta)$$

In these expressions d_i is the distance from the sensor to the ith point of intersection, i = 1, 2: $d_i = \sqrt{(U-X_i)^2 + (V-Y_i)^2}$

33. If $|\theta-\beta'| = 0°$ or $180°$, go to 36. If $|\theta-\beta'| = 90°$ or $270°$, go to 35.

34. Evaluate the target density from Y_1' to Y_2' along the line $y = mx + c$:

Let $Y_1' = \min(Y_1, Y_2)$

$Y_2' = \max(Y_1, Y_2)$

$\phi(z)$ = density function of a standard normal random variable evaluated at z

$$K = (m/\sqrt{\sigma_Y^2 + m^2\sigma_X^2})\ \phi(c/\sqrt{\sigma_Y^2 + m^2\sigma_X^2})$$

Subdivide the interval (Y_1', Y_2') into n segments of length h. At the midpoint of each segement, y_j, compute:

32. Calculate the median width of the ...

...elsipod ... the angular ...

$$hK\left(\sqrt{\sigma_Y^2 + m^2\sigma_X^2} / m\sigma_X\sigma_Y\right) \;\phi\left(\left(y_j - \frac{c\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}\right)\left(\sqrt{\sigma_Y^2 + m^2\sigma_X^2} / m\sigma_X\sigma_Y\right)\right)$$

$$\int_0^{360} \exp\left[-\frac{1}{2}\left(\frac{(st_L)^2\cos^2(\theta-\psi) - 2st_L y_j \;\cos(\theta-\psi)}{m^2\sigma_X^2} + \frac{(st_L)^2\sin^2(\theta-\psi) - 2st_L y_j \;\sin(\theta-\psi)}{\sigma_Y^2}\right)\right]\frac{d\psi}{360}$$

when the integral must be numerically evaluated.

Sum the results over all n segments.

Go to 37.

35. Evaluate the target density from Y_1' to Y_2' along the line $x = U$:

Let $Y_1' = \min(Y_1, Y_2)$

$Y_2' = \max(Y_1, Y_2)$

$\phi(z)$ = density function of standard normal random variable

$K = \dfrac{1}{\sigma_X}\;\phi\left(\dfrac{U}{\sigma_X}\right)$

Subdivide the interval (Y_1', Y_2') into n segments of length h. At the midpoint of each segment, y_j, compute:

$$hK\,\frac{1}{\sigma_X}\left(\phi\left(\frac{y_j}{\sigma_Y}\right)\right) \int_0^{360} \exp\left[-\frac{1}{2}\left(\frac{(st_L)^2\cos(\theta-\psi) - 2st_L U\cos(\theta-\psi)}{\sigma_X^2} + \frac{(st_L)^2\sin^2(\theta-\psi) - 2st_L y_j \sin(\theta-\psi)}{\sigma_Y^2}\right)\right]\frac{d\psi}{360}$$

where the integral must be numerically evaluated.

Sum the results over all n segments. Go to 37.

36. Evaluate the target density from X_1' to X_2' along the line y = V:

Let $X_1' = \min(X_1, X_2)$

$X_2' = \max(X_1, X_2)$

$\phi(z)$ = density of standard normal random variable

$$K = \frac{1}{\sigma_Y} \phi\left(\frac{V}{\sigma_Y}\right)$$

Subdivide the interval (X_1', X_2') into n segments of length h. At the midpoint of each segment, x_j, compute:

$$hK \frac{1}{\sigma_X} \left(\phi\left(\frac{x_j}{\sigma_X}\right)\right) \int_0^{360} \exp\left[-\frac{1}{2}\left(\frac{(st_L)^2\cos(\theta-\psi) - 2st_L x_j(\cos(\theta-\psi)}{\sigma_X^2} + \frac{(st_L)^2\sin^2(\theta-\psi) - 2st_L V\sin(\theta-\psi)}{\sigma_Y^2}\right)\right] \frac{d\psi}{360}$$

where the integral must be numerically evaluated. Sum results over all n segments.

37. Multiply the value of the target density just computed by W.

38. Multiply this result by

$$\frac{1}{\sigma_\beta} \phi\left(\frac{\beta' - \beta}{\sigma_\beta}\right)$$

where ϕ is the standard normal density function.

39. Let PROB = PROB+ (the results of the calculations in steps 20 through 38 for each if the I subintervals of size $\Delta\beta$).

40. If the fractional interval of size $\delta\Delta\beta$ on this side has been considered go to 41. Otherwise, let $\beta' = \frac{1}{2}\delta\Delta\beta + \frac{1}{2}\Delta\beta$. Repeat calculations 21 through 38 once. Let PROB = PROB+ (the result calculated at this step).

41. If the computations on both sides of β have been computed, go to 42. Otherwise repeat the computations from 18 to 40 on the other side of β by letting $\Delta\beta = -\Delta\beta$, $\delta\Delta\beta = -\delta\Delta\beta$.

42. Let $\beta' = \beta$. Repeat steps 21 through 38 once. Let PROB = PROB+ (the results of this calculation).

43. Let j = the number of discrete threat speeds to be considered. Let PROB1 = PROB1 + P[S=s]PROB. Go to 15.

44. Repeat steps 15 through 43 once with the sensor located at each of three points:

(1) $(U,V) = (u_o,v_o)$, the mean of the sensor density;

(2) $(U,V) = (u_o + .97\sigma\cos(\theta-\beta-90),$
$v_o + .97\sigma\sin(\theta-\beta-90))$;

(3) $(U,V) = (u_o - .97\cos(\theta-\beta-90),$
$v_o - .97\sin(\theta-\beta-90))$;

where σ is the standard deviation of the sensor density. The value of PROB1 calculated at each iteration will be weighted by the approximate probability that the sensor is

39. Let PROB = PROB + (the results of the ...

in steps 10 through 18 for each if the i sum ...

located in the region of the sensor error circle represented by the applicable value (U,V). If p_3 is of the order .86 or greater, multiply by 1/3.

45. The value of PROB2 calculated after completion of step 44 is the relative probability that the threat lies along bearing β at time t_L, given the threat was in $p_1 \times 100\%$ ellipse at time t_o, β_t is in (β_L, β_U) and the sensor is in a $p_3 \times 100\%$ confidence circle.

located in the region of the sensor ... circle ...

sented by the equation ...

CALCULATOR PROGRAM

PART I

000	76	LBL
001	11	A
002	94	+/-
003	85	+
004	01	1
005	95	=
006	23	LNX
007	65	X
008	02	2
009	94	+/-
010	95	=
011	34	√X
012	42	STO
013	02	02
014	35	1/X
015	42	STO
016	05	05
017	42	STO
018	06	06
019	91	R/S
020	76	LBL
021	12	B
022	42	STO
023	03	03
024	49	PRD
025	05	05
026	35	1/X
027	42	STO
028	10	10
029	91	R/S
030	76	LBL
031	13	C
032	42	STO
033	04	04
034	49	PRD
035	06	06
036	49	PRD
037	10	10
038	43	RCL
039	06	06
040	42	STO
041	07	07
042	33	X²
043	42	STO
044	08	08
045	43	RCL
046	05	05
047	49	PRD
048	07	07
049	33	X²
050	42	STO
051	09	09
052	91	R/S
053	76	LBL
054	14	D
055	42	STO
056	11	11
057	75	-
058	91	R/S
059	76	LBL
060	15	E
061	95	=
062	32	X:T
063	91	R/S
064	76	LBL
065	16	A'
066	94	+/-
067	32	X:T
068	37	P/R
069	42	STO
070	13	13
071	32	X:T
072	42	STO
073	12	12
074	00	0
075	42	STO
076	35	35
077	03	3
078	06	6
079	00	0
080	42	STO
081	14	14
082	89	π
083	65	X
084	02	2
085	95	=
086	34	√X
087	42	STO
088	15	15
089	91	R/S
090	76	LBL
091	17	B'
092	42	STO
093	16	16
094	68	NOP
095	65	X
096	91	R/S
097	76	LBL
098	18	C'
099	95	=

100	42	STO
101	17	17
102	42	STO
103	00	00
104	91	R/S
105	76	LBL
106	19	D'
107	42	STO
108	18	18
109	55	÷
110	02	2
111	95	=
112	42	STO
113	19	19
114	22	INV
115	44	SUM
116	00	00
117	43	RCL
118	18	18
119	22	INV
120	49	PRD
121	00	00
122	43	RCL
123	00	00
124	59	INT
125	85	+
126	01	1
127	95	=
128	42	STO
129	20	20
130	48	EXC
131	00	00
132	22	INV
133	59	INT
134	42	STO
135	21	21
136	43	RCL
137	18	18
138	49	PRD
139	21	21
140	02	2
141	22	INV
142	49	PRD
143	21	21
144	42	STO
145	01	01
146	91	R/S
147	76	LBL
148	10	E'
149	42	STO
150	22	22
151	91	R/S
152	42	STO
153	31	31
154	91	R/S
155	32	X:T
156	71	SBR
157	38	SIN
158	43	RCL
159	22	22
160	85	+
161	43	RCL
162	17	17
163	95	=
164	71	SBR
165	37	P/R
166	32	X:T
167	43	RCL
168	22	22
169	75	−
170	43	RCL
171	17	17
172	95	=
173	71	SBR
174	37	P/R
175	77	GE
176	02	02
177	15	15
178	32	X:T
179	42	STO
180	30	30
181	43	RCL
182	11	11
183	22	INV
184	77	GE
185	01	01
186	97	97
187	32	X:T
188	43	RCL
189	30	30
190	77	GE
191	01	01
192	95	95
193	01	1
194	91	R/S
195	02	2
196	91	R/S
197	85	+
198	01	1
199	08	8

200	00	0	250	43	RCL
201	95	=	251	14	14
202	22	INV	252	54	)
203	77	GE	253	22	INV
204	01	01	254	59	INT
205	93	93	255	65	×
206	32	X:T	256	43	RCL
207	43	RCL	257	14	14
208	30	30	258	54	)
209	77	GE	259	92	RTN
210	01	01	260	76	LBL
211	95	95	261	38	SIN
212	61	GTO	262	01	1
213	01	01	263	67	EQ
214	93	93	264	03	03
215	42	STO	265	40	40
216	30	30	266	02	2
217	43	RCL	267	22	INV
218	11	11	268	67	EQ
219	32	X:T	269	02	02
220	77	GE	270	73	73
221	01	01	271	86	STF
222	95	95	272	02	02
223	32	X:T	273	03	3
224	85	+	274	67	EQ
225	01	1	275	02	02
226	08	8	276	84	84
227	00	0	277	87	IFF
228	95	=	278	02	02
229	32	X:T	279	02	02
230	43	RCL	280	86	86
231	30	30	281	94	+/-
232	32	X:T	282	34	FX
233	77	GE	283	92	RTN
234	01	01	284	86	STF
235	95	95	285	03	03
236	61	GTO	286	53	(
237	01	01	287	93	.
238	93	93	288	09	9
239	76	LBL	289	07	7
240	37	P/R	290	65	×
241	53	(	291	43	RCL
242	53	(	292	31	31
243	53	(	293	65	×
244	24	CE	294	53	(
245	85	+	295	43	RCL
246	43	RCL	296	11	11
247	14	14	297	75	−
248	54	)	298	43	RCL
249	55	÷	299	22	22

300	75	-
301	09	9
302	00	0
303	54	)
304	39	COS
305	54	)
306	87	IFF
307	02	02
308	03	03
309	11	11
310	94	+/-
311	44	SUM
312	12	12
313	53	(
314	93	.
315	09	9
316	07	7
317	65	×
318	43	RCL
319	31	31
320	65	×
321	53	(
322	43	RCL
323	11	11
324	75	-
325	43	RCL
326	22	22
327	75	-
328	09	9
329	00	0
330	54	)
331	38	SIN
332	54	)
333	87	IFF
334	02	02
335	03	03
336	38	38
337	94	+/-
338	44	SUM
339	13	13
340	92	RTN

PART II - PROGRAM 1

000	43	RCL
001	22	22
002	42	STO
003	23	23
004	87	IFF
005	00	00
006	05	05
007	13	13
008	22	INV
009	97	DSZ
010	00	00
011	00	00
012	21	21
013	22	INV
014	86	STF
015	00	00
016	43	RCL
017	18	18
018	61	GTO
019	00	00
020	29	29
021	86	STF
022	00	00
023	43	RCL
024	19	19
025	85	+
026	43	RCL
027	21	21
028	95	=
029	44	SUM
030	23	23
031	43	RCL
032	14	14
033	44	SUM
034	23	23
035	22	INV
036	49	PRD
037	23	23
038	65	X
039	48	EXC
040	23	23
041	22	INV
042	59	INT
043	95	=
044	42	STO
045	23	23
046	94	+/-
047	75	-
048	43	RCL
049	11	11
050	95	=
051	99	PRT
052	42	STO
053	24	24
054	50	IXI
055	32	X:T
056	09	9
057	00	0
058	67	EQ
059	35	1/X
060	02	2
061	07	7
062	00	0
063	67	EQ
064	35	1/X
065	00	0
066	67	EQ
067	78	Σ+
068	01	1
069	08	8
070	00	0
071	22	INV
072	67	EQ
073	01	01
074	33	33
075	76	LBL
076	78	Σ+
077	43	RCL
078	13	13
079	50	IXI
080	32	X:T
081	43	RCL
082	04	04
083	22	INV
084	77	GE
085	00	00
086	04	04
087	86	STF
088	02	02
089	43	RCL
090	13	13
091	42	STO
092	29	29
093	42	STO
094	30	30
095	02	2
096	09	9
097	42	STO
098	31	31
099	04	4

100	42	STO
101	32	32
102	03	3
103	42	STO
104	33	33
105	71	SBR
106	45	YX
107	42	STO
108	28	28
109	94	+/-
110	42	STO
111	27	27
112	02	2
113	08	8
114	42	STO
115	38	38
116	02	2
117	07	7
118	42	STO
119	39	39
120	02	2
121	09	9
122	42	STO
123	34	34
124	05	5
125	42	STO
126	36	36
127	06	6
128	42	STO
129	24	24
130	61	GTO
131	02	02
132	91	91
133	22	INV
134	86	STF
135	03	03
136	22	INV
137	86	STF
138	02	02
139	43	RCL
140	24	24
141	30	TAN
142	42	STO
143	24	24
144	42	STO
145	36	36
146	43	RCL
147	12	12
148	94	+/-
149	49	PRD
150	36	36
151	43	RCL
152	13	13
153	44	SUM
154	36	36
155	43	RCL
156	36	36
157	33	X^2
158	32	X:T
159	43	RCL
160	03	03
161	33	X^2
162	65	×
163	43	RCL
164	24	24
165	33	X^2
166	85	+
167	43	RCL
168	04	04
169	33	X^2
170	95	=
171	42	STO
172	34	34
173	22	INV
174	77	GE
175	00	00
176	04	04
177	75	−
178	32	X:T
179	95	=
180	34	√X
181	65	×
182	43	RCL
183	10	10
184	95	=
185	42	STO
186	27	27
187	94	+/-
188	75	−
189	53	(
190	43	RCL
191	24	24
192	65	×
193	43	RCL
194	36	36
195	54	)
196	22	INV
197	44	SUM
198	27	27
199	95	=

```
200   42  STO
201   28   28
202   43  RCL
203   03   03
204   33  X²
205   55   ÷
206   43  RCL
207   34   34
208   95   =
209   49  PRD
210   27   27
211   49  PRD
212   28   28
213   43  RCL
214   27   27
215   42  STO
216   29   29
217   43  RCL
218   28   28
219   42  STO
220   30   30
221   43  RCL
222   24   24
223   49  PRD
224   29   29
225   49  PRD
226   30   30
227   43  RCL
228   36   36
229   44  SUM
230   29   29
231   44  SUM
232   30   30
233   61  GTO
234   02   02
235   91   91
236   76  LBL
237   35  1/X
238   43  RCL
239   12   12
240   50  I×I
241   32  X↕T
242   43  RCL
243   03   03
244   22  INV
245   77   GE
246   00   00
247   04   04
248   86  STF
249   03   03
250   43  RCL
251   12   12
252   42  STO
253   27   27
254   42  STO
255   28   28
256   02   2
257   07   7
258   42  STO
259   31   31
260   03   3
261   42  STO
262   32   32
263   04   4
264   42  STO
265   33   33
266   71  SBR
267   45  YX
268   42  STO
269   30   30
270   94  +/-
271   42  STO
272   29   29
273   03   3
274   00   0
275   42  STO
276   38   38
277   02   2
278   09   9
279   42  STO
280   39   39
281   02   2
282   07   7
283   42  STO
284   34   34
285   06   6
286   42  STO
287   36   36
288   05   5
289   42  STO
290   24   24
291   02   2
292   07   7
293   42  STO
294   31   31
295   02   2
296   09   9
297   42  STO
298   32   32
299   71  SBR
```

300	28	LOG
301	42	STO
302	37	37
303	02	2
304	08	8
305	42	STO
306	31	31
307	03	3
308	00	0
309	42	STO
310	32	32
311	71	SBR
312	28	LOG
313	44	SUM
314	37	37
315	43	RCL
316	19	19
317	22	INV
318	87	IFF
319	00	00
320	03	03
321	24	24
322	43	RCL
323	21	21
324	50	I×I
325	30	TAN
326	50	I×I
327	49	PRD
328	37	37
329	87	IFF
330	02	02
331	67	EQ
332	87	IFF
333	03	03
334	67	EQ
335	43	RCL
336	36	36
337	65	×
338	43	RCL
339	08	08
340	55	÷
341	53	(
342	43	RCL
343	08	08
344	85	+
345	43	RCL
346	09	09
347	65	×
348	43	RCL
349	24	24
350	33	X²
351	54	)
352	42	STO
353	38	38
354	95	=
355	22	INV
356	44	SUM
357	29	29
358	22	INV
359	44	SUM
360	30	30
361	43	RCL
362	38	38
363	34	√X
364	49	PRD
365	29	29
366	49	PRD
367	30	30
368	43	RCL
369	24	24
370	50	I×I
371	65	×
372	43	RCL
373	07	07
374	95	=
375	22	INV
376	49	PRD
377	29	29
378	22	INV
379	49	PRD
380	30	30
381	43	RCL
382	29	29
383	32	X:T
384	43	RCL
385	30	30
386	77	GE
387	03	03
388	92	92
389	42	STO
390	29	29
391	32	X:T
392	36	PGM
393	19	19
394	12	B
395	75	−
396	43	RCL
397	29	29
398	36	PGM
399	19	19

400	12	B
401	95	=
402	65	×
403	53	(
404	43	RCL
405	36	36
406	55	÷
407	43	RCL
408	38	38
409	34	√X
410	54	)
411	36	PGM
412	19	19
413	11	A
414	55	÷
415	43	RCL
416	38	38
417	34	√X
418	65	×
419	43	RCL
420	24	24
421	50	IXI
422	65	×
423	43	RCL
424	37	37
425	65	×
426	53	(
427	53	(
428	53	(
429	01	1
430	08	8
431	00	0
432	32	X:T
433	53	(
434	43	RCL
435	23	23
436	75	-
437	43	RCL
438	22	22
439	54	)
440	22	INV
441	77	GE
442	04	04
443	48	48
444	75	-
445	43	RCL
446	14	14
447	54	)
448	65	×
449	43	RCL
450	16	16
451	55	÷
452	43	RCL
453	17	17
454	54	)
455	36	PGM
456	19	19
457	11	A
458	54	)
459	65	×
460	43	RCL
461	16	16
462	55	÷
463	43	RCL
464	17	17
465	95	=
466	61	GTO
467	77	GE
468	76	LBL
469	67	EQ
470	73	RC*
471	36	36
472	22	INV
473	64	PD*
474	38	38
475	22	INV
476	64	PD*
477	39	39
478	73	RC*
479	38	38
480	36	PGM
481	19	19
482	12	B
483	75	-
484	73	RC*
485	39	39
486	36	PGM
487	19	19
488	12	B
489	95	=
490	65	×
491	53	(
492	73	RC*
493	34	34
494	55	÷
495	73	RC*
496	24	24
497	54	)
498	36	PGM
499	19	19

```
500  11   A
501  55   ÷
502  73  RC*
503  24   24
504  95   =
505  61  GTO
506  04   04
507  22   22
508  76  LBL
509  77   GE
510  44  SUM
511  35   35
512  99  PRT
513  87  IFF
514  01   01
515  79   x̄
516  22  INV
517  87  IFF
518  00   00
519  00   00
520  04   04
521  22  INV
522  97  DSZ
523  01   01
524  89   π
525  22  INV
526  86  STF
527  00   00
528  01   1
529  94  +/-
530  49  PRD
531  18   18
532  49  PRD
533  19   19
534  49  PRD
535  21   21
536  43  RCL
537  20   20
538  42  STO
539  00   00
540  61  GTO
541  00   00
542  00   00
543  76  LBL
544  89   π
545  86  STF
546  01   01
547  22  INV
548  86  STF
549  00   00
550  25  CLR
551  42  STO
552  23   23
553  43  RCL
554  22   22
555  61  GTO
556  00   00
557  29   29
558  76  LBL
559  79   x̄
560  22  INV
561  86  STF
562  01   01
563  43  RCL
564  35   35
565  99  PRT
566  91  R/S
567  76  LBL
568  45  YX
569  53   (
570  53   (
571  01   1
572  75   -
573  53   (
574  73  RC*
575  31   31
576  55   ÷
577  73  RC*
578  32   32
579  54   )
580  33  X²
581  54   )
582  34  √X
583  65   ×
584  73  RC*
585  33   33
586  54   )
587  92  RTN
588  76  LBL
589  28  LOG
590  53   (
591  53   (
592  73  RC*
593  31   31
594  75   -
595  43  RCL
596  12   12
597  54   )
598  33  X²
599  85   +
```

```
600  53  (
601  73 RC*
602  32  32
603  75  -
604  43 RCL
605  13  13
606  54  )
607  33 X²
608  54  )
609  34 ΓX
610  92 RTN
```

PART II PROGRAM 2

```
000  43  RCL        050  95   =
001  22   22        051  99  PRT
002  42  STO        052  42  STO
003  23   23        053  24   24
004  87  IFF        054  50  I×I
005  00   00        055  32  X:T
006  05   05        056  09   9
007  13   13        057  00   0
008  22  INV        058  67   EQ
009  97  DSZ        059  35  1/X
010  00   00        060  02   2
011  00   00        061  07   7
012  21   21        062  00   0
013  22  INV        063  67   EQ
014  86  STF        064  35  1/X
015  00   00        065  00   0
016  43  RCL        066  67   EQ
017  18   18        067  78  Σ+
018  61  GTO        068  01   1
019  00   00        069  08   8
020  29   29        070  00   0
021  86  STF        071  22  INV
022  00   00        072  67   EQ
023  43  RCL        073  01   01
024  19   19        074  33   33
025  85   +         075  76  LBL
026  43  RCL        076  78  Σ+
027  21   21        077  43  RCL
028  95   =         078  13   13
029  44  SUM        079  50  I×I
030  23   23        080  32  X:T
031  43  RCL        081  43  RCL
032  14   14        082  04   04
033  44  SUM        083  22  INV
034  23   23        084  77   GE
035  22  INV        085  00   00
036  49  PRD        086  04   04
037  23   23        087  86  STF
038  65   ×         088  02   02
039  48  EXC        089  43  RCL
040  23   23        090  13   13
041  22  INV        091  42  STO
042  59  INT        092  29   29
043  95   =         093  42  STO
044  42  STO        094  30   30
045  23   23        095  02   2
046  94  +/-        096  09   9
047  85   +         097  42  STO
048  43  RCL        098  31   3
049  11   11        099  04   4
```

100	42	STO
101	32	32
102	03	3
103	42	STO
104	33	33
105	71	SBR
106	45	Y^X
107	42	STO
108	28	28
109	94	+/-
110	42	STO
111	27	27
112	02	2
113	08	8
114	42	STO
115	38	38
116	02	2
117	07	7
118	42	STO
119	39	39
120	02	2
121	09	9
122	42	STO
123	34	34
124	05	5
125	42	STO
126	36	36
127	06	6
128	42	STO
129	24	24
130	61	GTO
131	02	02
132	91	91
133	22	INV
134	86	STF
135	00	00
136	22	INV
137	86	STF
138	02	02
139	43	RCL
140	24	24
141	30	TAN
142	42	STO
143	24	24
144	42	STO
145	36	36
146	43	RCL
147	12	12
148	94	+/-
149	49	PRD
150	36	36
151	43	RCL
152	13	13
153	44	SUM
154	36	36
155	43	RCL
156	36	36
157	33	X^2
158	32	$X \rightleftharpoons T$
159	43	RCL
160	03	03
161	33	X^2
162	65	×
163	43	RCL
164	24	24
165	33	X^2
166	85	+
167	43	RCL
168	04	04
169	33	X^2
170	95	=
171	42	STO
172	34	34
173	22	INV
174	77	GE
175	00	00
176	04	04
177	75	-
178	32	$X \rightleftharpoons T$
179	95	=
180	34	$\sqrt{X}$
181	65	×
182	43	RCL
183	10	10
184	95	=
185	42	STO
186	27	27
187	94	+/-
188	75	-
189	53	(
190	43	RCL
191	24	24
192	65	×
193	43	RCL
194	36	36
195	54	)
196	22	INV
197	44	SUM
198	27	27
199	95	=

```
200   42  STO
201   28   28
202   43  RCL
203   03   03
204   33  X²
205   55  ÷
206   43  RCL
207   34   34
208   95  =
209   49  PRD
210   27   27
211   49  PRD
212   28   28
213   43  RCL
214   27   27
215   42  STO
216   29   29
217   43  RCL
218   28   28
219   42  STO
220   30   30
221   43  RCL
222   24   24
223   49  PRD
224   29   29
225   49  PRD
226   30   30
227   43  RCL
228   36   36
229   44  SUM
230   29   29
231   44  SUM
232   30   30
233   61  GTO
234   02   02
235   91   91
236   76  LBL
237   35  1/X
238   43  RCL
239   12   12
240   50  I×I
241   32  X:T
242   43  RCL
243   03   03
244   22  INV
245   77   GE
246   00   00
247   04   04
248   86  STF
249   03   03
```

```
250   43  RCL
251   12   12
252   42  STO
253   27   27
254   42  STO
255   28   28
256   02   2
257   07   7
258   42  STO
259   31   31
260   03   3
261   42  STO
262   32   32
263   04   4
264   42  STO
265   33   33
266   71  SBR
267   45  YX
268   42  STO
269   30   30
270   94  +/-
271   42  STO
272   29   29
273   03   3
274   00   0
275   42  STO
276   33   33
277   02   2
278   09   9
279   42  STO
280   39   39
281   02   2
282   07   7
283   42  STO
284   34   34
285   06   6
286   42  STO
287   36   36
288   05   5
289   42  STO
290   24   24
291   02   2
292   07   7
293   42  STO
294   31   31
295   02   2
296   09   9
297   42  STO
298   32   32
299   71  SBR
```

```
300  28 LOG        350  33 X²
301  42 STO        351  65  ×
302  37  37        352  43 RCL
303  02  2         353  09  09
304  08  8         354  54  )
305  42 STO        355  42 STO
306  31  31        356  38  38
307  03  3         357  95  =
308  00  0         358  44 SUM
309  42 STO        359  27  27
310  32  32        360  44 SUM
311  71 SBR        361  28  28
312  28 LOG        362  43 RCL
313  44 SUM        363  38  38
314  37  37        364  34 √X
315  43 RCL        365  55  ÷
316  19  19        366  43 RCL
317  22 INV        367  07  07
318  87 IFF        368  95  =
319  00  00        369  49 PRD
320  03  03        370  27  27
321  24  24        371  49 PRD
322  43 RCL        372  28  28
323  21  21        373  43 RCL
324  50 I×I        374  27  27
325  30 TAN        375  32 X:T
326  50 I×I        376  43 RCL
327  49 PRD        377  28  28
328  37  37        378  77  GE
329  87 IFF        379  03  03
330  02  02        380  84  84
331  67  EQ        381  42 STO
332  87 IFF        382  27  27
333  03  03        383  32 X:T
334  67  EQ        384  36 PGM
335  43 RCL        385  19  19
336  24  24        386  12  B
337  65  ×         387  75  -
338  43 RCL        388  43 RCL
339  36  36        389  27  27
340  65  ×         390  36 PGM
341  43 RCL        391  19  19
342  09  09        392  12  B
343  55  ÷         393  95  =
344  53  (         394  65  ×
345  43 RCL        395  53  (
346  08  08        396  43 RCL
347  85  +         397  36  36
348  43 RCL        398  55  ÷
349  24  24        399  43 RCL
```

```
400  38   38
401  34  ΓX
402  54   )
403  36  PGM
404  19   19
405  11   A
406  55   ÷
407  43  RCL
408  38   38
409  34  ΓX
410  65   ×
411  43  RCL
412  37   37
413  65   ×
414  68  NOP
415  68  NOP
416  68  NOP
417  68  NOP
418  68  NOP
419  68  NOP
420  68  NOP
421  68  NOP
422  68  NOP
423  68  NOP
424  68  NOP
425  68  NOP
426  53   (
427  53   (
428  53   (
429  01   1
430  08   8
431  00   0
432  32  X:T
433  53   (
434  43  RCL
435  23   23
436  75   -
437  43  RCL
438  22   22
439  54   )
440  22  INV
441  77   GE
442  04   04
443  48   48
444  75   -
445  43  RCL
446  14   14
447  54   )
448  65   ×
449  43  RCL
```

```
450  16   16
451  55   ÷
452  43  RCL
453  17   17
454  54   )
455  36  PGM
456  19   19
457  11   A
458  54   )
459  65   ×
460  43  RCL
461  16   16
462  55   ÷
463  43  RCL
464  17   17
465  95   =
466  61  GTO
467  77   GE
468  76  LBL
469  67   EQ
470  73  RC*
471  36   36
472  22  INV
473  64  PD*
474  38   38
475  22  INV
476  64  PD*
477  39   39
478  73  RC*
479  38   38
480  36  PGM
481  19   19
482  12   B
483  75   -
484  73  RC*
485  39   34
486  36  PGM
487  19   19
488  12   B
489  95   =
490  65   ×
491  53   (
492  73  RC*
493  34   34
494  55   ÷
495  73  RC*
496  24   24
497  54   )
498  36  PGM
499  19   19
```

```
500  11  A
501  55  ÷
502  73  RC*
503  24  24
504  95  =
505  61  GTO
506  04  04
507  10  10
508  76  LBL
509  77  GE
510  44  SUM
511  35  35
512  99  PRT
513  87  IFF
514  01  01
515  79  x̄
516  22  INV
517  87  IFF
518  00  00
519  00  00
520  04  04
521  22  INV
522  97  DSZ
523  01  01
524  89  π
525  22  INV
526  86  STF
527  00  00
528  01  1
529  94  +/-
530  49  PRD
531  18  18
532  49  PRD
533  19  19
534  49  PRD
535  21  21
536  43  RCL
537  20  20
538  42  STO
539  00  00
540  61  GTO
541  00  00
542  00  00
543  76  LBL
544  89  π
545  86  STF
546  01  01
547  22  INV
548  86  STF
549  00  00
550  25  CLR
551  42  STO
552  23  23
553  43  RCL
554  22  22
555  61  GTO
556  00  00
557  29  29
558  76  LBL
559  79  x̄
560  22  INV
561  86  STF
562  01  01
563  43  RCL
564  35  35
565  99  PRT
566  91  R/S
567  76  LBL
568  45  Y^X
569  53  (
570  53  (
571  01  1
572  75  -
573  53  (
574  73  RC*
575  31  31
576  55  ÷
577  73  RC*
578  32  32
579  54  )
580  33  X²
581  54  )
582  34  √X
583  65  ×
584  73  RC*
585  33  33
586  54  )
587  92  RTN
588  76  LBL
589  28  LOG
590  53  (
591  53  (
592  73  RC*
593  31  31
594  75  -
595  43  RCL
596  12  12
597  54  )
598  33  X²
599  65  ×
```

```
000  76 LBL
001  11  A
002  65  X
003  53  (
004  03  3
005  35 1/X
006  54  )
007  42 STO
008  00  00
009  95  =
010  42 STO
011  01  01
012  91 R/S
013  76 LBL
014  12  B
015  65  X
016  43 RCL
017  00  00
018  95  =
019  44 SUM
020  01  01
021  91 R/S
022  76 LBL
023  13  C
024  65  X
025  43 RCL
026  00  00
027  95  =
028  44 SUM
029  01  01
030  43 RCL
031  01  01
032  99 PRT
033  76 LBL
034  14  D
035  22 INV
036  49 PRD
037  01  01
038  43 RCL
039  01  01
040  99 PRT
041  91 R/S
042  76 LBL
043  15  E
044  22 INV
045  49 PRD
046  01  01
047  43 RCL
048  01  01
049  99 PRT
050  91 R/S
```

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